Exact Foldy-Wouthuysen transformation for gravitational waves and magnetic field background

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Abstract

We consider an exact Foldy-Wouthuysen transformation for the Dirac spinor field on the combined background of a gravitational wave and constant uniform magnetic field. By taking the classical limit of the spinor field Hamiltonian we arrive at the equations of motion for the non-relativistic spinning particle. Two different kinds of the gravitational fields are considered and in both cases the effect of the gravitational wave on the spinor field and on the corresponding spinning particle may be enforced by the sufficiently strong magnetic field. This result can be relevant for the astrophysical applications and, in principle, useful for creating the gravitational wave detectors based on atomic physics and precise interferometry.

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I. INTRODUCTION

The study of the gravitational effects on quantum-mechanical systems represents an important issue, since all the physical objects, on both macroscopic and microscopic scales, are indeed located in a curved spacetime or in non-inertial reference frames. This subject is currently of a special interest (see, e.g., an overview [1]). Despite the weakness of the gravitational interaction, its effects were actually observed at the quantum-mechanical level. In particular, one can mention the famous Colella-Overhauser-Werner (COW) [2] and Bonse-Wroblewski [3] experiments, in which the quantum-mechanical phase shift due to the gravitational and inertial forces was measured, thereby verifying the validity of the equivalence principle for the non-relativistic neutron waves.

Among the different configurations of the gravitational field, the case of the gravitational waves appears to be especially interesting [4]. This is related to the fact that considerable efforts are applied to the experimental search of the gravitational waves (see, e.g., [5]). Therefore, it may be important to achieve better understanding of the behavior of the quantum fields and particles under the action of the gravitational wave.

In this paper, we study the dynamics of the Dirac particle in a plane gravitational wave. The plane-fronted gravitational waves represent an important class of exact solutions which generalize the basic properties of electromagnetic waves in flat spacetime to the case of curved spacetime geometry. The relevant investigation of the gravitational waves in general relativity has a long and rich history, see, e.g., [6]. On the other hand, the approximate plane gravitational wave solution, that arises from the linearized Einstein's field equations, is usually used for the analysis of the various physical aspects related to the radiation, propagation, and detection of the gravitational waves. We will consider both cases, the approximate weak wave and the exact nonlinear plane-fronted wave.

The general framework for the Dirac theory in curved spacetime was developed in many publications, see the early reference [7], and the overview and the reference given in a recent paper [8]. We will use the notation and conventions of the latter work.

The paper is organized as follows. In section II we construct the universal form of an exact Foldy-Wouthuysen transformation for Dirac spinors and apply this scheme to the case of the background approximate plane gravitational wave and electromagnetic field. Several simpler particular (previously well known) cases are considered in Appendix. In section III we use the result of the previous section to consider the particle Hamiltonian and corresponding equations of motion for the non-relativistic spinning particle interacting to the combined background of gravitational wave and electromagnetic field. In section IV we provide the comparison with the exact gravitational wave case and in the last section V we draw our conclusions.

II. LINEAR GRAVITATIONAL PERTURBATIONS AND UNIVERSAL FORM OF AN EXACT FOLDY-WOUTHUYSEN TRANSFORMATION

Let us start from the relatively simple case of the usual (linear) gravitational waves. Using this case as an example, we shall also develop the general form of the exact Foldy-Wouthuysen transformation which can be applied to many previously explored cases and also to the more complicated case of the exact plane-fronted gravitational wave.

The metric of the weak gravitational wave reads (see, e.g. [9])

$$g_{ij} = \eta_{ij} + h_{ij}\,, \tag{1}$$

where $\eta_{ij} = \mathrm{diag}(-1, 1, 1, 1)$ is the flat Minkowski metric and the nonzero components of the gravitational perturbation $h_{\mu\nu}$ are (in the Cartesian local coordinates $x^i = (t, x, y, z)$)

$$h_{yy} = -h_{zz} = -2v\,, \qquad h_{yz} = h_{zy} = -2u\,. \tag{2}$$

Here $v = v(ct-x)$ and $u = u(ct-x)$ are the two functions which describe a wave propagating along the x axis. It is assumed that the functions $v = v(ct-x)$ and $u = u(ct-x)$ are small such that the linear approximation in these functions is valid. As usual, the gravitational wave can have 2 polarization states, and each of the functions v and u correspond to one of the possible polarizations.

For the formulation of the Dirac theory in curved spacetime, we need the tetrad fields. The coframe 1-form reads:

$$\vartheta^0 = cdt, \qquad \vartheta^1 = dx, \tag{3}$$
$$\vartheta^2 = (1+v)dy + udz, \tag{4}$$
$$\vartheta^3 = (1-v)dz + udy. \tag{5}$$

The corresponding inverse vector frame is described by

$$e_0 = \frac{1}{c}\partial_t, \qquad e_1 = \partial_x, \tag{6}$$
$$e_2 = (1-v)\partial_y - u\partial_z, \tag{7}$$
$$e_3 = (1+v)\partial_z - u\partial_y. \tag{8}$$

It is straightforward to construct the Riemannian connection (the Christoffel symbols). The nonzero components of the local connection 1-forms read:

$${\Gamma_2}^0 = {\Gamma_0}^2 = v'\,\vartheta^2 + u'\,\vartheta^3, \tag{9}$$
$${\Gamma_3}^0 = {\Gamma_0}^3 = u'\,\vartheta^2 - v'\,\vartheta^3, \tag{10}$$
$${\Gamma_2}^1 = -{\Gamma_1}^2 = v'\,\vartheta^2 + u'\,\vartheta^3, \tag{11}$$
$${\Gamma_3}^1 = -{\Gamma_1}^3 = u'\,\vartheta^2 - v'\,\vartheta^3. \tag{12}$$

The primes denote the derivatives w.r.t. the argument of the functions: $v' = dv(z)/dz, u' = du(z)/dz$.

One can check that the Cartan structure equation is fulfilled. Indeed, we have $d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta = 0$, which demonstrates that this is the torsion-free Riemannian connection.

Let us construct the Dirac operator on the background of the metric described above. The spinor covariant derivative is

$$D_\alpha = e_\alpha \rfloor D, \qquad D := d + \frac{i}{4}\,\widehat{\sigma}^{\alpha\beta}\,\Gamma_{\alpha\beta}. \tag{13}$$

We have $\widehat{\sigma}^{\alpha\beta} = i\gamma^{[\alpha}\gamma^{\beta]}$, and consequently,

$$\frac{i}{4}\,\widehat{\sigma}^{\alpha\beta}\,\Gamma_{\alpha\beta} = -\frac{1}{2}\left(\gamma^2\gamma^0\,\Gamma_{20} + \gamma^3\gamma^0\,\Gamma_{30} + \gamma^1\gamma^2\,\Gamma_{12} + \gamma^3\gamma^1\,\Gamma_{31}\right). \tag{14}$$

We shall use the standard [10] representation for the gamma matrices:

$$\gamma^0 = \beta = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix}, \qquad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \quad a = 1, 2, 3, \tag{15}$$

where the Pauli matrices are:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16}$$

The alpha matrices and the spin matrix are defined, as usual, by the relations

$$\overrightarrow{\alpha} = \gamma^0\overrightarrow{\gamma} = \begin{pmatrix} 0 & \overrightarrow{\sigma} \\ \overrightarrow{\sigma} & 0 \end{pmatrix}, \qquad \overrightarrow{\Sigma} = \begin{pmatrix} \overrightarrow{\sigma} & 0 \\ 0 & \overrightarrow{\sigma} \end{pmatrix}. \tag{17}$$

Using these definitions, and substituting (9)-(12), we find

$$\frac{i}{4}\,\widehat{\sigma}^{\alpha\beta}\,\Gamma_{\alpha\beta} = \frac{1}{2}\left(\alpha^2 + i\,\Sigma^3\right)\left(v'\,\vartheta^2 + u'\,\vartheta^3\right) + \frac{1}{2}\left(\alpha^3 - i\,\Sigma^2\right)\left(u'\,\vartheta^2 - v'\,\vartheta^3\right). \tag{18}$$

The Dirac operator is constructed as $\gamma^\mu D_\mu$, and correspondingly, we obtain

$$\begin{aligned}\gamma^\mu e_\mu\rfloor\left(\frac{i}{4}\,\widehat{\sigma}^{\alpha\beta}\,\Gamma_{\alpha\beta}\right) \;&=\; \frac{v'}{2}\left[\left(\gamma^2\alpha^2 - \gamma^3\alpha^3\right) + i\left(\gamma^2\Sigma^3 + \gamma^3\Sigma^2\right)\right] \\ &\quad + \frac{u'}{2}\left[\left(\gamma^2\alpha^3 + \gamma^3\alpha^2\right) + i\left(\gamma^3\Sigma^3 - \gamma^2\Sigma^2\right)\right]. \end{aligned} \tag{19}$$

A direct computation shows that

$$\gamma^2\alpha^2 - \gamma^3\alpha^3 = \gamma^2\alpha^3 + \gamma^3\alpha^2 = \gamma^2\Sigma^3 + \gamma^3\Sigma^2 = \gamma^3\Sigma^3 - \gamma^2\Sigma^2 = 0. \tag{20}$$

Hence, all the terms on the r.h.s. of the equation (19) do vanish . As a result, the contribution of the spinor connection drops out completely and the Dirac operator is reduced just to the

partial derivative terms

$$\begin{aligned}\gamma^\mu D_\mu = \gamma^\mu e_\mu \rfloor d &= \frac{1}{c}\,\gamma^0\,\partial_t + \gamma^1\,\partial_x + \Big[(1-v)\,\gamma^2 - u\,\gamma^3\Big]\,\partial_y \\ &\quad + \Big[(1+v)\,\gamma^3 - u\,\gamma^2\Big]\,\partial_z \\ &= \beta\Big\{\frac{1}{c}\,\partial_t + \alpha^1\,\partial_x + \Big[(1-v)\,\alpha^2 - u\,\alpha^3\Big]\,\partial_y \\ &\quad + \Big[(1+v)\,\alpha^3 - u\,\alpha^2\Big]\,\partial_z\Big\}. \end{aligned} \tag{21}$$

Now we are in the position to consider the covariant Dirac equation

$$(i\hbar\gamma^\mu D_\mu - mc)\,\psi = 0\,. \tag{22}$$

After we recast it into the familiar Schrödinger form, this equation reduces to

$$i\hbar\frac{\partial\psi}{\partial t} = \widehat{\mathcal{H}}\,\psi \tag{23}$$

with the Hamilton operator of the form

$$\widehat{\mathcal{H}} = mc^2\beta + c\alpha^1\,p^1 + \Big[(1-v)\,c\alpha^2 - u\,c\alpha^3\Big]\,p^2 + \Big[(1+v)\,c\alpha^3 - u\,c\alpha^2\Big]\,p^3\,, \tag{24}$$

where p^1, p^2, p^3 are components of the momentum vector $\overrightarrow{p}$.

The crucial observation is that this form of the Hamiltonian falls into the class of models which admit the anticommuting involution operator, cf. [8, 11, 12],

$$J = i\gamma_5\beta. \tag{25}$$

The latter is Hermitian, $J^\dagger = J$, and unitary, $JJ^\dagger = J^2 = 1$, and it anticommutes both with the Hamiltonian and with the β matrix:

$$J\widehat{\mathcal{H}} + \widehat{\mathcal{H}}J = 0, \qquad J\beta + \beta J = 0. \tag{26}$$

Consequently, the exact FW transformation can be constructed in this case.

Moreover, if one introduces the (in the simplest case, constant and uniform) magnetic field, this important feature is not destroyed, since the only thing which is technically needed is to replace the operators of momentum $\overrightarrow{p}$ by the expressions $\overrightarrow{p} - \frac{e}{c}\overrightarrow{A}$. After performing this operation we arrive at the following Hamiltonian:

$$\begin{aligned} H &= \beta mc^2 - i\hbar c\Big[\alpha^1(\partial_x + \frac{e}{i\hbar c}A_x) \\ &+ (\alpha^2 - v\alpha^2 - u\alpha^3)(\partial_y + \frac{e}{i\hbar c}A_y) + (\alpha^3 - u\alpha^2 + v\alpha^3)(\partial_z + \frac{e}{i\hbar c}A_z)\Big]\,. \end{aligned} \tag{27}$$

For a while we do not impose restrictions for the vector potential. Let us rewrite the expression (27) using new notations, which prove useful in what follows [13]

$$H = \beta mc^2 + \alpha^b K^a_b \partial_a + \alpha^a g_a\,, \tag{28}$$

where

$$K^a_b = -i\hbar c \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - v - u\alpha^2\alpha^3 & 0 \\ 0 & 0 & 1 + v - u\alpha^3\alpha^2 \end{pmatrix} , \tag{29}$$

$$g_a = \, -e\, (A_1\, , \, A_2 - vA_2 - uA_3\, , \, A_3 - uA_2 + vA_3)\, , \tag{30}$$

According to the standard prescription [11] (the structure of exact FW for the scalar fields with non-minimal coupling to gravity has been discussed recently in [14]) the fist step in deriving the exact FW is to calculate the square of the Hamiltonian H^2. In order to accomplish this, we define, additionally, conjugated quantities

$$\overline{K^a_b} = -i\hbar c \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - v + u\alpha^2\alpha^3 & 0 \\ 0 & 0 & 1 + v + u\alpha^3\alpha^2 \end{pmatrix} , \tag{31}$$

$$\overline{g_a} \, = \, (\alpha^1 g_1\, , \, \alpha^2 g_2\, , \, \alpha^3 g_3\,)\, , \tag{32}$$

and

$$\overline{\epsilon^{abc}} = \, (\alpha_1 \epsilon^{a1c}\, , \, \alpha_2 \epsilon^{a2c}\, , \, \alpha_3 \epsilon^{a3c})\, . \tag{33}$$

Using these notations, a direct calculation gives, after some algebra, the following square of the Hamiltonian H^2:

$$\begin{aligned} H^2 \, &= \, m^2c^4 \, + \, \overline{K^{ac}} K^b_c \partial_a \partial_b \, + \, \overline{K^a_b} g^b \partial_a + g^b K^a_b \partial_a \, + \, g^2 \\ &+ \, \overline{K^{ac}} (\partial_a K^b_c) \partial_b \, + \, i\Sigma_d \epsilon^{bcd} \overline{K^a_b} (\partial_a K^e_c) \partial_e \, + \, i\Sigma_d \overline{\epsilon^{bcd}} K^a_b \overline{g_c} \partial_a \\ &- \, i\epsilon^{bcd} \Sigma_d K^a_b g_c \partial_a \, + \, i\Sigma_d \overline{\epsilon^{bcd}} K^a_b (\partial_a \overline{g_c}) \, + \, \overline{K^a_b} (\partial_a g^b)\, . \end{aligned} \tag{34}$$

Here the partial derivatives inside the parenthesis act only on the content of these parenthesis. Let us notice that the components of the matrix K^a_b depend exclusively on the metric while the components of the vector g_a depend on both the metric and the electromagnetic potential A_a. The expression (34) is rather general and may be used not only for the linear gravitational waves but also in various other particular cases. In order to illustrate this fact, we consider several known cases in the Appendix.

III. PARTICLE HAMILTONIAN AND EQUATIONS OF MOTION

Starting from the equation (34), for the sake of simplicity, we consider only one polarization of the gravitational wave. Namely, we choose $u = 0$. Then, we obtain

$$K^a_b = \overline{K^a_b} = -i\hbar c(\delta^a_b + T^a_b v) \qquad , \qquad g_b = -e(\delta^a_b + T^a_b v) A_a\, , \tag{35}$$

where

$$T^a_b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,, \tag{36}$$

and there is no need anymore to use the notations $\overline{g_b}$ and $\overline{\epsilon^{abc}}$. Thus we arrive at the Hamiltonian

$$\begin{aligned} H \,\simeq\, & \frac{1}{2mc^2}\,\beta\,(\delta^{ab}+2T^{ab}v)\,\Big[(cp_a - eA_a)(cp_b - eA_b) - e\hbar c\,\varepsilon_{cad}\Sigma^d\partial^c(A_b)\Big] \\ + \, & \frac{\hbar}{2mc}\,\beta\,\varepsilon^{abc}\partial_a(v)\Sigma_c T^d_b(cp_d - eA_d) + \beta mc^2\,. \end{aligned} \tag{37}$$

A. Nonrelativistic limit

The next step is to present the Dirac fermion ψ in the form

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{-imc^2t/\hbar} \quad , \tag{38}$$

and use the equation

$$i\hbar\partial_t\psi = H\psi \tag{39}$$

to derive the Hamiltonian for the 2-spinor φ. Inserting (38) into (39), we obtain the two-component equation

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \Big(-mc^2 + H\Big)\begin{pmatrix} \varphi \\ \chi \end{pmatrix}\,. \tag{40}$$

Using the fact that Hamiltonian is an even function, we obtain, in the φ sector, the Hamiltonian

$$\begin{aligned} H \,=\, & \frac{1}{2mc^2}\,(\delta^{ab}+2T^{ab}v)\,\Big[(cp_a - eA_a)(cp_b - eA_b) - e\hbar c\varepsilon_{cad}\sigma^d\partial^c A_b\Big] \\ + \, & \frac{\hbar}{2mc}\varepsilon^{abc}\partial_a(v)\sigma_c T^d_b(cp_d - eA_d)\,. \end{aligned} \tag{41}$$

The r.h.s. of the last equation (41) is proportional to $1/m$, that is the same level of the nonrelativistic approximation which one meets in the case of Pauli equation. Therefore one can expect that the same equation can be obtained starting from the original eq. (28). This calculation represents an efficient check of our results, so it is worthwhile to perform it now. Let us apply the standard procedure for deriving the Pauli equation (see, e.g. [15]). Using the representation (40) in (28) and (39) we apply the low-energy regime , that is assume that the term mc^2 is dominant ($|mc^2\chi| \gg |i\hbar\partial_t\chi|$). In this way we arrive at the equation

$$i\hbar\frac{\partial}{\partial t}\varphi = \frac{1}{2mc^2}(\sigma^b K^a_b\partial_a + \sigma^a g_a)(\sigma^d K^c_d\partial_c + \sigma^c g_c)\varphi\,. \tag{42}$$

After some transformations, the r.h.s. of the last equation coincides with the r.h.s. of our eq. (41).

It is easy to check that, if the gravitational wave is absent $v \equiv 0$, one obtains

$$i\hbar\frac{\partial}{\partial t}\varphi = \frac{1}{2mc^2}[\vec{\sigma}(c\vec{p} - e\vec{A}) \cdot \vec{\sigma}(c\vec{p} - e\vec{A})]\varphi \ , \tag{43}$$

and finally, after a some algebra, a usual Pauli equation

$$i\hbar\frac{\partial}{\partial t}\varphi = [\frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2 - \frac{e\hbar}{2mc}\vec{\sigma} \cdot \vec{B}]\varphi \ . \tag{44}$$

B. Nonrelativistic spinor particle on the background of the gravitational wave and electromagnetic field

One can use the the result (37) for deriving the equation for the nonrelativistic spinor particle. Let us follow [16, 17], where the similar calculation has been performed for the nonrelativistic equation on the background of torsion field. The same problem has been treated in [18] starting from the perturbative FW transformation.

The Hamiltonian operador $\hat{H}$ corresponding to the energy (37) is constructed in terms of the operators $\hat{x}_a$, $\hat{p}_a$ and $\hat{\sigma}_a$. The equations of motion have the form

$$i\hbar\frac{d\hat{x}_a}{dt} = [\hat{x}_a, H]\,, \qquad i\hbar\frac{d\hat{p}_a}{dt} = [\hat{p}_a, H]\,, \qquad i\hbar\frac{d\hat{\sigma}_a}{dt} = [\hat{\sigma}_a, H]\ . \tag{45}$$

In order to achieve the nonrelativistic limit one has to calculate the commutators of the operators in (45) and take the limit $\hbar \to 0$. Disrearding the $\mathcal{O}(\hbar)$ terms, we do not need to care about the ordering of the operators, and then finally we arrive at the semiclassical equations of motion for the spinning particle on the background of the gravitational field and constant magnetic field. The result can be presented using the notation $A'_a = T_{ab}A^b$. Then the equations become

$$\frac{dx_a}{dt} = \frac{1}{m}(\delta_{ab} + 2\,T_{ab}\,v)\left(p^b - \frac{e}{c}A^b\right)\ , \tag{46}$$

$$\frac{dp_a}{dt} = -\frac{1}{m}T^{bc}\,\partial_a v\,\left(p_b - \frac{e}{c}A_b\right)\left(p_c - \frac{e}{c}\,A_c\right) + (\delta_{bc} + 2T_{bc}v)\,\left(p^b - \frac{e}{c}A^b\right)\,\frac{e}{mc}\,\partial_a A^c\,, \tag{47}$$

$$\frac{d\sigma_a}{dt} = \frac{e}{mc}\,\varepsilon_{abc}\sigma^c\left[\vec{B} + 2v\,\mathrm{rot}\,(\vec{A}')\right]^b - \frac{1}{mc}\,(cp^d - eA^d)\,\sigma^e\,\left[T_{ed}(\partial_a v) - T_{ad}(\partial_e v)\right]\ . \tag{48}$$

Let us notice that the first equation here demonstrates deviation from the usual relation between the Lagrangian velocity and the canonically conjugated momenta. The difference is due to the gravitational wave. Taken together, the first two equations define the coordinate dependence of the particle. Using these two equations, one can construct the analog of

Lorentz force in the presence of gravitational wave

$$
\begin{aligned}
m\,\ddot{x}_a \;=\; & \frac{e}{c}\,(\delta_{ab}+2T_{ab}v)\big[\dot{\vec{x}}\times\vec{B}\,]^b - \frac{e}{c}(\delta_{ab}+2T_{ab}v)\frac{\partial A^b}{\partial t} \\
+ \; & 2mT_{ab}\frac{dv}{dt}\dot{x}^b - mT_{bc}\,\dot{x}^b\,\dot{x}^c(\partial_a v)\,. \qquad (49)
\end{aligned}
$$

The last equation in (48) in describing the spin dynamics of the particle. It is remarkable that this equation depends on the velocity. Let us notice that the same property holds for the spinning particle in the external torsion field for both nonrelativistic [16] and relativistic [19] cases.

Perhaps the spin dynamics of the particle is the most interesting result here. In order to understand this point, let us make the following observation. It is easy to notice that the last equation in (48) describes spin precession even for the case when the magnetic field is absent. The effect is due to the presence of the matrix T^a_b defined in eq.(36) and the $\mathrm{rot}\,\mathbf{A}'$. The last vector is distinct from the magnetic field and maybe nonzero when $\mathbf{B} = \mathrm{rot}\,\mathbf{A} = 0$. Therefore, there is a possibility to observe the spin precession without magnetic field and this can, in principle, become a new basis for the gravitational waves detector of the new type. We expect to consider this issue in more details in the near future. A detailed analysis of the definitions of the momentum and spin dynamical operators (along the lines recently done for the static gravitational field [20]) will be required.

C. Vanishing electromagnetic field

In the absence of the electromagnetic potential, $A_a = 0$, the equations of motion can be intergrated. With $x^a = (x, y, z)$, the second and the third equations in (49) read

$$
\frac{d}{dt}\,\dot{y} \;=\; -\,2\dot{y}\,\frac{dv}{dt}, \qquad (50)
$$

$$
\frac{d}{dt}\,\dot{z} \;=\; 2\dot{z}\,\frac{dv}{dt}. \qquad (51)
$$

These equations are easily integrated, yielding the components of the velocities as the functions of v:

$$
\dot{y} = Y\,e^{-2v}, \qquad \dot{z} = Z\,e^{2v}\,, \qquad (52)
$$

with the integration constants Y and Z. The integration of the equation for x is slightly more nontrivial. This equation reads

$$
\ddot{x} = (\dot{y}^2 - \dot{z}^2)\,\partial_x v = f(v)\,\partial_x v, \qquad (53)
$$

with $f(v) = Y^2 e^{-4v} - Z^2 e^{4v}$. Recalling that $v = v(\xi)$ with $\xi = ct - x$, we notice that $\partial_x v = -\,dv/d\xi$, and furthermore $\ddot{\xi} = -\ddot{x}$. Accordingly, we can recast (53) into the equation for ξ:

$$
2\ddot{\xi} = -\,\frac{dU}{d\xi}, \qquad U = \frac{1}{2}\left(Y^2 e^{-4v} + Z^2 e^{4v}\right). \qquad (54)
$$

Multiplying with $\dot{\xi}$, we find the first integral

$$\dot{\xi}^2 + U(\xi) = I, \tag{55}$$

and the solution $\xi = \xi(t)$ is then obtained in quadratures

$$\int \frac{d\xi}{\sqrt{I - U(\xi)}} = t - t_0 \,. \tag{56}$$

The explicit form of the solution $\xi = \xi(t)$ of course depends on the integration constant I and on the explicit form of the wave function $v = v(\xi)$ that will determine the "potential" $U(\xi)$. For the harmonic wave, for example, $v = v_0 \cos\xi$, and then $U(\xi) = \tilde{Y}^2 e^{-4\cos\xi} + \tilde{Z}^2 e^{4\cos\xi}$ (with some new constants $\tilde{Y}$ and $\tilde{Z}$). After finding $\xi(t)$, we can use it for the final integration of the first order equations (52). Thus, we finally obtain the coordinates of the particle as functions of time $y(t)$, $z(t)$ and $x(t) = ct - \xi(t)$.

In order to complete the analysis of the particle dynamics, we have to solve the equation for the spin. In the absence of the electromagnetic field, it reads:

$$\dot{\sigma}_a = \frac{\sqrt{p_y^2 + p_z^2}\,(\partial_x v)}{m}\, M_{ab}\, \sigma_b, \tag{57}$$

with the matrix

$$M_{ab} = \begin{pmatrix} 0 & \pi_y & -\pi_z \\ -\pi_y & 0 & 0 \\ \pi_z & 0 & 0 \end{pmatrix}. \tag{58}$$

Here $\pi_y = p_y/\sqrt{p_y^2 + p_z^2}$, $\pi_z = p_z/\sqrt{p_y^2 + p_z^2}$. Note that $\dot{p_y} = \dot{p_z} = 0$, see (47). The integration is then straightforward, yielding the final result for the dynamics of the spin

$$\overrightarrow{\sigma} \;=\; \exp\left[s(t)M\right] \cdot \overrightarrow{\sigma}_0 \,, \qquad \text{where} \qquad s(t) = \frac{\sqrt{p_y^2 + p_z^2}}{m} \int_{t_0}^{t} dt\, \partial_x(v)\,. \tag{59}$$

Note that since for the cubic term we have $M^3 = -M$, the matrix exponential actually contains only the terms M and M^2, and it can be written explicitly as

$$e^{s(t)M} = \mathbf{1} + M \sin s + M^2 \left(1 - \cos s\right) \,.$$

The solutions (52), (56), (59) show that the nonrelativistic spinning particle has very peculiar behaviour in the field of the weak gravitational wave.

IV. COMPARISON WITH EXACT GRAVITATIONAL WAVE

The exact plane-fronted gravitational wave, in the simplest case is described in the Cartesian local coordinates $x^i = (t, x, y, z)$ by the line element $ds^2 = g_{ij}dx^i dx^j$ with the metric (see [6, 21, 22])

$$g_{ij} = \eta_{ij} + h_{ij} \,, \tag{60}$$

where the nonzero components are

$$h_{tt} = h_{xx} = -U, \qquad h_{tx} = h_{xt} = U \tag{61}$$

expressed in terms of a function $U(\xi, y, z)$. It can depend arbitrarily on $\xi = ct - x$, and is a harmonic function in the two last variables, i.e. $\Delta_{(2)}U = (\partial^2_{yy} + \partial^2_{zz})U = 0$. The coframe reads

$$\vartheta^0 = \left(1 + \frac{U}{2}\right) cdt - \frac{U}{2}\, dx, \tag{62}$$

$$\vartheta^1 = \frac{U}{2}\, cdt + \left(1 - \frac{U}{2}\right) dx, \tag{63}$$

$$\vartheta^2 = dy, \qquad \vartheta^3 = dz. \tag{64}$$

The inverse frame is easily found:

$$e_0 = \left(1 - \frac{U}{2}\right) \frac{1}{c}\partial_t - \frac{U}{2}\, \partial_x, \tag{65}$$

$$e_1 = \frac{U}{2c}\, \partial_t + \left(1 + \frac{U}{2}\right) \partial_x, \tag{66}$$

$$e_2 = \partial_y, \qquad e_3 = \partial_z. \tag{67}$$

From this we can verify straightforwardly that

$$\frac{i}{4}\, \widehat{\sigma}^{\alpha\beta}\, \Gamma_{\alpha\beta} = \left(\gamma^0 - \gamma^1\right) \frac{1}{4} \left(\gamma^2\partial_y U + \gamma^3\partial_z U\right) \left(\vartheta^1 - \vartheta^0\right). \tag{68}$$

Accordingly, we find that

$$\gamma^\mu e_\mu\rfloor \left(\frac{i}{4}\, \widehat{\sigma}^{\alpha\beta}\, \Gamma_{\alpha\beta}\right) = -\left(\gamma^0 - \gamma^1\right)^2 \frac{1}{4} \left(\gamma^2\partial_y U + \gamma^3\partial_z U\right) = 0, \tag{69}$$

since $(\gamma^0 - \gamma^1)^2 \equiv 0$. Thus, just like in the case of an approximate wave, the spinor connection term drops out completely from the Dirac equation.

Consequently, the Dirac operator has the form

$$\gamma^\alpha D_\alpha = \gamma^\alpha e_\alpha\rfloor d = \frac{1}{c}\Big[\gamma^0 + \frac{U}{2}(\gamma^1 - \gamma^0)\Big]\partial_t + \Big[\gamma^1 + \frac{U}{2}(\gamma^1 - \gamma^0)\Big]\partial_x + \gamma^2\partial_y + \gamma^3\partial_z\,. \tag{70}$$

It proves very useful, before we go to the exact FW transformation, we make a Lorentz transformation of the coframe $\vartheta^\alpha \longrightarrow \vartheta'^\alpha = \Lambda^\alpha{}_\beta\vartheta^\beta$, using the matrix (written in the 2×2 block form with $\mathbf{0}$ and $\mathbf{1}$ as the 2×2 zero and unit matrices, respectively)

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} L & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad \text{where} \qquad L = \frac{1}{\sqrt{1-U}} \begin{pmatrix} 1 - U/2 & U/2 \\ U/2 & 1 - U/2 \end{pmatrix}. \tag{71}$$

Under this transformation, the tetrad frame is changed to $e_\alpha \rightarrow e'_\alpha = (\Lambda^{-1})^\beta{}_\alpha e_\beta$, whereas the local Lorentz connection transforms to $\Gamma'_\alpha{}^\beta = \Lambda^\beta{}_\nu\Gamma_\mu{}^\nu(\Lambda^{-1})^\mu{}_\alpha + \Lambda^\beta{}_\gamma d(\Lambda^{-1})^\gamma{}_\alpha$. One can then verify that the transformed spinor connection term do contribute to the Dirac operator

$$\gamma^\mu e'_\mu\rfloor \left(\frac{i}{4}\, \widehat{\sigma}^{\alpha\beta}\, \Gamma'_{\alpha\beta}\right) = \frac{U'}{4(1-U)^{\frac{3}{2}}}(\gamma^0 - \gamma^1) + \frac{1}{4(1-U)}\, \gamma^0 \left(\gamma^2\gamma^1\partial_y U + \gamma^3\gamma^1\partial_z U\right), \tag{72}$$

where $U' = dU/d\xi$. Another contribution comes from the ordinary derivative term

$$\gamma^\alpha e'_\alpha \rfloor d = \gamma^0 \left(\frac{\sqrt{1-U}}{c} \partial_t - \frac{U}{\sqrt{1-U}} \partial_x \right) + \gamma^1 \frac{1}{\sqrt{1-U}} \partial_x + \gamma^2 \partial_y + \gamma^3 \partial_z \,. \tag{73}$$

Collecting all together, we can finally write the Dirac equation (22) in the Schrödinger form (23). As a last step we perform the rescaling transformation [7] of the wave function

$$\Psi' = (1-U)^{\frac{1}{4}} \Psi \,.$$

In this way we arrive at the final form of the Hamiltonian that is explicitly Hermitian:

$$\begin{aligned} H' \;=\; & \beta m c^2 V + c p_x - \frac{c}{2}(1-\alpha^1)\left(V^2 p_x + p_x V^2\right) \\ & + \frac{c}{2}\left[\alpha^2 (V p_y + p_y V) + \alpha^3 (V p_z + p_z V)\right] \\ & - \frac{i\hbar c}{2}\, \alpha^1 \left[\alpha^2 \partial_y (V) + \alpha^3 \partial_z (V)\right] \,. \end{aligned} \tag{74}$$

Here $V = 1/\sqrt{1-U}$.

Let us notice that switching the gravitational wave off with $U = 0$ (hence $V = 1$), we recover the Hamiltonian of the free particle. The Hamiltonian (74) is not anticommuting with the matrix $J = i\gamma^5\beta$. As a result, the condition for performing the exact FW transformation ($\{J\,,\, H'\} = 0$) is not satisfied in this case.

V. DISCUSSION AND CONCLUSION

In this paper, we have derived an exact Foldy-Wouthuysen transformation for the Dirac spinor field on the combined background of a gravitational wave and constant uniform magnetic field. The motivation for the presence of the magnetic field is to check the possibility of the amplification of the influence of the gravitational wave on a Dirac particle. According to our calculations (52), (47) and (48) such an effect is possible. This result can be relevant for the astrophysical applications and, in principle, could be useful for improving the gravitational wave detectors based on atomic physics and precise interferometry (see, e.g., recent discussion in [23]). The actual dynamics of the Dirac particle in a plane gravitational wave will be studied separately.

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APPENDIX A: FREE PARTICLE

Let us consider several simple known particular cases when the general formula (34) can be applied.

For the case of a free particle, we have $K^a_b = -i\,\hbar\,c\,\delta^a_b$ and $g_a = 0$, because $u = v = 0$ and $A_b = 0$. Then

$$H = \beta\sqrt{m^2c^4 + c^2p^2}\,, \tag{A1}$$

that is a well-known known textbooks result.

APPENDIX B: PARTICLE IN A MAGNETIC FIELD

In this case $K^a_b = -i\,\hbar\,c\,\delta^a_b$ and $g_a = -eA_a$, because $u = v = 0$. The expression for the Hamiltonian is

$$H = \beta\sqrt{m^2c^4 + (c\overrightarrow{p} - e\overrightarrow{A})^2 - \hbar ce\,\overrightarrow{\Sigma}.\overrightarrow{B}}\,. \tag{B1}$$

This is exactly the result obtained by Eriksen and Kolsrud [11] for this case.

APPENDIX C: PARTICLE WITH ANOMALOUS MAGNETIC MOMENT IN A STATIC MAGNETIC FIELD

In this case we start from the Hamiltonian with

$$K^a_b = -i\,\hbar\,c\,\delta^a_b$$

and $g_a = -eA_a + \alpha_a\mu_I\,\overrightarrow{\Sigma}.\overrightarrow{B}$, and again with $u = v = 0$.

This version is a bit more complicated because one has to account for the commutators of g_a with α^b and β. After some calculations we arrive at the following form of the Hamitonian

$$H^2 = m^2c^4 + (c\overrightarrow{p} - e\overrightarrow{A})^2 - 2\mu_I mc^2\overrightarrow{\Sigma}.\overrightarrow{B} + \mu_I^2B^2 + \mu_I\beta\overrightarrow{\Sigma}\cdot\left(\overrightarrow{B}\times\overrightarrow{p} - \overrightarrow{p}\times\overrightarrow{B}\right)\,. \tag{C1}$$

Once again, this expression is in a perfect agreement with the result obtained by Eriksen and Kolsrud [11].

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